



Chen, J., Sun, H., Woodruff, D., & Zhang, Q. (2016). *Communication-optimal distributed clustering*. Poster session presented at Annual Conference on Neural Information Processing .

Peer reviewed version

[Link to publication record in Explore Bristol Research](#)
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Communication-Optimal Distributed Clustering

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Abstract

Clustering large datasets is a fundamental problem with a number of applications in machine learning. Data is often collected on different sites and clustering needs to be performed in a distributed manner with low communication. We would like the quality of the clustering in the distributed setting to match that in the centralized setting for which all the data resides on a single server. In this work, we study both graph and geometric clustering problems in two distributed models: (1) a point-to-point model, and (2) a model with a broadcast channel. We give protocols in both models which we show are nearly optimal by proving almost matching communication lower bounds. Our work highlights the surprising power of a broadcast channel for clustering problems; roughly speaking, to cluster n points or n vertices in a graph distributed across s servers, for a worst-case partitioning the communication complexity in a point-to-point model is $n \cdot s$, while in the broadcast model it is $n + s$. We implement our algorithms and demonstrate this phenomenon on real life datasets, showing that our algorithms are also very efficient in practice.

1 Introduction

Clustering is a fundamental task in machine learning with widespread applications in data mining, computer vision, and social network analysis. Example applications of clustering include grouping similar webpages by search engines, finding users with common interests in a social network, and identifying different objects in a picture or video. For these applications, one can model the objects that need to be clustered as points in Euclidean space \mathbb{R}^d , where the similarities of two objects are represented by the Euclidean distance between the two points. Then the task of clustering is to choose k points as centers, so that the total distance between all input points to their corresponding closest center is minimized. Depending on different distance objective functions, three typical problems have been studied: k -means, k -median, and k -center.

The other popular approach for clustering is to model the input data as vertices of a graph, and the similarity between two objects is represented by the weight of the edge connecting the corresponding vertices. For this scenario, one is asked to partition the vertices into clusters so that the “highly connected” vertices belong to the same cluster. A widely-used approach for graph clustering is *spectral clustering*, which embeds the vertices of a graph into the points in \mathbb{R}^k through the bottom k eigenvectors of the graph’s Laplacian matrix, and applies k -means on the embedded points.

Both the spectral clustering and the geometric clustering algorithms mentioned above have been widely used in practice, and have been the subject of extensive theoretical and experimental studies over the decades. However, these algorithms are designed for the centralized setting, and are not applicable in the setting of large-scale datasets that are maintained remotely by different sites. In particular, collecting the information from all the remote sites and performing a centralized clustering algorithm is infeasible due to high communication costs, and new distributed clustering algorithms with low communication cost need to be developed.

There are several natural communication models, and we focus on two of them: (1) a point-to-point model, and (2) a model with a broadcast channel. In the former, sometimes referred to as the *message-passing model*, there is a communication channel between each pair of users. This may be impractical,

and the so-called *coordinator model* can often be used in place; in the coordinator model there is a centralized server called the coordinator, and all communication goes through the coordinator. This affects the total communication by a factor of two, since the coordinator can forward a message from one server to another and therefore simulate a point-to-point protocol. There is also an additional additive $O(\log s)$ bits per message, where s is the number of servers, since a server must specify to the coordinator where to forward its message. In the model with a broadcast channel, sometimes referred to as the *blackboard model*, the coordinator has the power to send a single message which is received by all s servers at once. This can be viewed as a model for single-hop wireless networks.

In both models we study the total number of bits communicated among all servers. Although the blackboard model is at least as powerful as the message-passing model, it is often unclear how to exploit its power to obtain better bounds for specific problems. Also, for a number of problems the communication complexity is the same in both models, such as computing the sum of s length- n bit vectors modulo two, where each server holds one bit vector [20], or estimating large moments [22]. Still, for other problems like set disjointness it can save a factor of s in the communication [5].

Our Contributions. We present algorithms for graph clustering: for any n -vertex graph whose edges are arbitrarily partitioned across s sites, our algorithms have communication cost $\tilde{O}(ns)$ in the message passing model, and have communication cost $\tilde{O}(n + s)$ in the blackboard model, where the \tilde{O} notation suppresses polylogarithmic factors. The algorithm in the message passing model has each server send a *spectral sparsifier* of its local data to the coordinator, who then merges them in order to obtain a spectral sparsifier of the union of the datasets, which is sufficient for solving the graph clustering problem. Our algorithm in the blackboard model is technically more involved, as we show a particular recursive sampling procedure for building a spectral sparsifier can be efficiently implemented using a broadcast channel. It is unclear if other natural ways of building spectral sparsifiers can be implemented with low communication in the blackboard model. Our algorithms demonstrate the surprising power of the blackboard model for clustering problems. Since our algorithms compute sparsifiers, they also have applications to solving symmetric diagonally dominant linear systems in a distributed model. Any such system can be converted into a system involving a Laplacian (see, e.g., [1]), from which a spectral sparsifier serves as a good preconditioner.

Next we show that $\Omega(ns)$ bits of communication is necessary in the message passing model to even recover a constant fraction of a cluster, and $\Omega(n + s)$ bits of communication is necessary in the blackboard model. This shows the optimality of our algorithms up to poly-logarithmic factors.

We then study clustering problems in constant-dimensional Euclidean space. We show for any $c > 1$, computing a c -approximation for k -median, k -means, or k -center correctly with constant probability in the message passing model requires $\Omega(sk)$ bits of communication. We then strengthen this lower bound, and show even for *bicriteria* clustering algorithms, which may output a constant factor more clusters and a constant factor approximation, our $\Omega(sk)$ bit lower bound still holds. Our proofs are based on communication and information complexity. Our results imply that existing algorithms [3] for k -median and k -means with $\tilde{O}(sk)$ bits of communication, as well as the folklore parallel guessing algorithm for k -center with $\tilde{O}(sk)$ bits of communication, are optimal up to poly-logarithmic factors. For the blackboard model, we present an algorithm for k -median and k -means that achieves an $O(1)$ -approximation using $\tilde{O}(s + k)$ bits of communication. This again separates the models.

We give empirical results which show that using spectral sparsifiers preserves the quality of spectral clustering surprisingly well in real-world datasets. For example, when we partition a graph with over 70 million edges (the *Sculpture* dataset) into 30 sites, only 6% of the input edges are communicated in the blackboard model and 8% are communicated in the message passing model, while the values of the normalized cut (the objective function of spectral clustering) given in those two models are at most 2% larger than the one given by the centralized algorithm, and the visualized results are almost identical. This is strong evidence that spectral sparsifiers can be a powerful tool in practical, distributed computation. When the number of sites is large, the blackboard model incurs significantly less communication than the message passing model, e.g., in the *Twomoons* dataset when there are 90 sites, the message passing model communicates 9 times as many edges as communicated in the blackboard model, illustrating the strong separation between these models that our theory predicts.

Related Work. There is a rich literature on spectral and geometric clustering algorithms from various aspects (see, e.g., [2, 18, 19, 21]). Balcan et al. [3, 4] and Feldman et al. [10] study distributed k -means ([3] also studies k -median), and present provable guarantees on the clustering quality.

Cohen et al. [7] study dimensionality reduction techniques for the input data matrices that can be used for distributed k -means. The main takeaway from previous work is that there is no previous work which develops protocols for spectral clustering in the common message passing and blackboard models, and lower bounds are lacking as well. For geometric clustering, while upper bounds exist (e.g., [3, 4, 10]), no provable lower bounds in either model existed, and our main contribution is to show that previous algorithms are optimal. We also develop a new protocol in the blackboard model.

2 Preliminaries

Let $G = (V, E, w)$ be an undirected graph with n vertices, m edges, and weight function $V \times V \rightarrow \mathbb{R}_{\geq 0}$, where every edge's weight is bounded by a polynomial of n . The set of neighbors of a vertex u is represented by $N(u)$, and its degree is $d_u = \sum_{u \sim v} w(u, v)$. The maximum degree of G is defined to be $\Delta(G) = \max_v \{d_v\}$. For any set $S \subseteq V$, let $\mu(S) \triangleq \sum_{u \in S} d_u$. For any sets $S, T \subseteq V$, we define $w(S, T) \triangleq \sum_{u \in S, v \in T} w(u, v)$ to be the total weight of edges crossing S and T . For two sets X and Y , the symmetric difference of X and Y is defined as $X \Delta Y \triangleq (X \setminus Y) \cup (Y \setminus X)$.

For any matrix $A \in \mathbb{R}^{n \times n}$, let $\lambda_1(A) \leq \dots \leq \lambda_n(A)$ be the eigenvalues of A , and let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ be the maximum and minimum eigenvalues of A . For any two matrices $A, B \in \mathbb{R}^{n \times n}$, we write $A \preceq B$ to represent $B - A$ is positive semi-definite (PSD). Notice that this condition implies that $x^\top A x \leq x^\top B x$ for any $x \in \mathbb{R}^n$. Sometimes we also use a weaker notation $(1 - \varepsilon)A \preceq_r B \preceq_r (1 + \varepsilon)A$ to indicate that $(1 - \varepsilon)x^\top A x \leq x^\top B x \leq (1 + \varepsilon)x^\top A x$ for all x in the row span of A .

Graph Laplacian. The Laplacian matrix of G is an $n \times n$ matrix L_G defined by $L_G = D_G - A_G$, where A_G is the adjacency matrix of G defined by $A_G(u, v) = w(u, v)$, and D_G is the $n \times n$ diagonal matrix with $D_{u,u} = \sum_{u \sim v} w(u, v)$ for any $u \in V[G]$. Alternatively, we can write L_G with respect to a *signed edge-vertex incidence matrix*: we assign every edge $e = \{u, v\}$ an arbitrary orientation, and let $B_G(e, v) = 1$ if v is e 's head, $B_G(e, v) = -1$ if v is e 's tail, and $B_G(e, v) = 0$ otherwise. We further define a diagonal matrix $W_G \in \mathbb{R}^{m \times m}$, where $W_G(e, e) = w_e$. Then, we can write L_G as $L_G = B_G^\top W_G B_G$. The *normalized Laplacian matrix* of G is defined by $\mathcal{L}_G \triangleq D_G^{-1/2} L_G D_G^{-1/2} = I - D_G^{-1/2} A_G D_G^{-1/2}$. We sometimes drop the subscript G when the underlying graph is clear from the context.

Spectral sparsification. For any graph $G = (V, E, w)$ with n vertices, we call a subgraph H of G with proper reweighting of the edges is a $(1 + \varepsilon)$ -spectral sparsifier if

$$(1 - \varepsilon)L_G \preceq L_H \preceq (1 + \varepsilon)L_G, \quad (1)$$

By definition, it is easy to show that, if we decompose the edge set of a graph $G = (V, E)$ into E_1, \dots, E_ℓ for a constant ℓ and H_i is a spectral sparsifier of $G_i = (V, E_i)$ for any $1 \leq i \leq \ell$, then the graph formed by the union of edge sets from H_i is a spectral sparsifier of G . It is known that, for any undirected graph G of n vertices, there is a $(1 + \varepsilon)$ -spectral sparsifier of G with $O(n/\varepsilon^2)$ edges, and it can be constructed in near-linear time [14]. We will show that a spectral sparsifier preserves the cluster structure of a graph.

Models of computation. We will study distributed clustering in two models for distributed data: the message passing model and the blackboard model. The message passing model represents those distributed computation systems with point-to-point communication, and the blackboard model represents those where messages can be broadcasted to all parties.

More precisely, in the message passing model there are s sites $\mathcal{P}_1, \dots, \mathcal{P}_s$, and one coordinator. These sites can talk to the coordinator through a two-way private channel. In fact, this is referred to as the coordinator model in Section 1, where it is shown to be equivalent to the point-to-point model up to small factors. The input is initially distributed at the s sites. The computation is in terms of rounds: at the beginning of each round, the coordinator sends a message to some of the s sites, and then each of those sites that have been contacted by the coordinator sends a message back to the coordinator. At the end, the coordinator outputs the answer. In the alternative blackboard model, the coordinator is simply a blackboard where these s sites $\mathcal{P}_1, \dots, \mathcal{P}_s$ can share information; in other words, if one site sends a message to the coordinator/blackboard then all the other $s - 1$ sites can see this information without further communication. The order for the sites to speak is decided by the contents of the blackboard.

For both models we measure the *communication cost* as the total number of bits sent through the channels. The two models are now standard in multiparty communication complexity (see, e.g., [5, 20, 22]). They are similar to the congested clique model [15] studied in the distributed computing community; the main difference is that in our models we do not post any bandwidth limitations at each channel but instead consider the total number of bits communicated.

3 Distributed graph clustering

In this section we study distributed graph clustering. We assume that the input graph $G = (V, E)$ has n vertices, and consists of k clusters. For any set $S \subseteq V$ with $\mu(S) \leq \mu(V)/2$, we define the conductance of set S by $\phi_G(S) \triangleq w(S, V \setminus S)/\mu(S)$, and the k -way expansion constant of graph G is defined by $\rho(k) \triangleq \min_{\text{partition } A_1, \dots, A_k} \max_{1 \leq i \leq k} \phi_G(A_i)$. Informally, we call S a *cluster* if vertices in S are highly connected to each other, and there are fewer edges between S and $V \setminus S$, i.e., set S has low conductance $\phi_G(S)$. Moreover, we say G has k clusters if we can partition V into S_1, \dots, S_k such that every S_i has low conductance $\phi_G(S_i)$, i.e., G has a small value of $\rho(k)$. While it is NP-hard to compute $\rho(k)$ and a partition S_1, \dots, S_k achieving $\rho(k)$, it is known that $\rho(k)$ is closely related to $\lambda_k(\mathcal{L}_G)$ through the following higher-order Cheeger inequality [13]:

$$\frac{\lambda_k(\mathcal{L}_G)}{2} \leq \rho(k) \leq O(k^2) \sqrt{\lambda_k(\mathcal{L}_G)}. \quad (2)$$

Based on (2), a large gap between $\lambda_{k+1}(\mathcal{L}_G)$ and $\rho(k)$ implies (i) existence of a k -way partition $\{S_i\}_{i=1}^k$ with bounded $\phi_G(S_i) \leq \rho(k)$, and (ii) any $(k+1)$ -way partition of G contains a subset with significantly high conductance $\rho(k+1) \geq \lambda_{k+1}(\mathcal{L}_G)/2$. That is, a large gap between $\lambda_{k+1}(\mathcal{L}_G)$ and $\rho(k)$ implies that G has *exactly* k clusters. In the following, we assume that G satisfies $\Upsilon \triangleq \lambda_{k+1}(\mathcal{L}_G)/\rho(k) = \Omega(k^3)$, and this condition guarantees that G has exactly k clusters. We use S_1, \dots, S_k to express a partition that achieves $\rho(k)$. We remark that the same assumption has been used in the literature for studying graph clustering problems [19].

Both algorithms presented in the section are based on the following *spectral clustering* algorithm: (i) compute the bottom k eigenvectors f_1, \dots, f_k of \mathcal{L}_G ; (ii) embed every vertex v to a point in \mathbb{R}^k through the embedding $F(v) = (f_1(v), \dots, f_k(v))$, with an appropriate normalization, and run k -means on the embedded points $\{F(v)\}_{v \in V}$.

3.1 The message passing model

We assume that the input graph $G = (V, E)$ has n vertices and m edges, and these m edges are arbitrarily allocated among s sites $\mathcal{P}_1, \dots, \mathcal{P}_s$. We further use E_i to denote the edge set maintained by site \mathcal{P}_i . Our proposed algorithm consists of two steps: (i) every site \mathcal{P}_i computes a linear-sized $(1 + \Theta(1))$ -spectral sparsifier $H_i = (V, E'_i)$ of $G_i \triangleq (V, E_i)$, and sends H_i to the coordinator; (ii) the coordinator runs a spectral clustering algorithm on the union of received graphs $H \triangleq (V, \cup_{i=1}^s E'_i)$. The following theorem summarizes the performance of this algorithm, and the approximation guarantee of our algorithm is as good as the provable guarantee of spectral clustering known in the non-distributed setting.

Theorem 3.1. *Let $G = (V, E)$ be an n -vertex graph that satisfies $\Upsilon = \Omega(k^3)$, and the edges of G are arbitrarily allocated among s sites. Then, there is a distributed algorithm that returns a partition A_1, \dots, A_k satisfying $\text{vol}(A_i \triangle S_i) = O(k^3 \cdot \Upsilon^{-1} \cdot \text{vol}(S_i))$ for any $1 \leq i \leq k$. The total communication cost of this algorithm is $O(ns)$ words.*

Proof. By the definition of the Laplacian matrix, we have that $L_G = \sum_{i=1}^s L_{G_i}$. Since every H_i is a $(1 + \Theta(1))$ -spectral sparsifier of graph G_i , we have that $(1 - \Theta(1))L_{H_i} \preceq L_{G_i} \preceq (1 + \Theta(1))L_{H_i}$. This implies that $(1 - \Theta(1))L_H \preceq L_G \preceq (1 + \Theta(1))L_H$, by the definition of H_i and graph Laplacians. Now we show that the our assumption of Υ preserves in H . By Lemma B.1 in Appendix B.1, we have for any $1 \leq i \leq k$ that $\phi_H(S_i) \in (\frac{1}{2}, 2) \phi_G(S_i)$, which implies that S_i has low conductance in H , and $\rho_H(k) \in (\frac{1}{2}, 2) \rho_G(k)$. To show that $\lambda_k(\mathcal{L}_H)$ is a constant approximation of $\lambda_k(\mathcal{L}_G)$, notice that $(1 - \Theta(1)) \cdot x^\top L_G x \leq x^\top L_H x \leq (1 + \Theta(1)) \cdot x^\top L_G x$ holds for any $x \in \mathbb{R}^n$. Hence it holds for any $x \in \mathbb{R}^n$ that $(1 - \varepsilon) \cdot x^\top D_G^{-1/2} L_G D_G^{-1/2} x \leq x^\top D_G^{-1/2} L_H D_G^{-1/2} x \leq (1 + \varepsilon) \cdot x^\top D_G^{-1/2} L_G D_G^{-1/2} x$.

Since $D_G^{-1/2} L_G D_G^{-1/2} = \mathcal{L}_G$ and $\frac{1}{2} D_G^{-1} \preceq D_H^{-1} \preceq 2 D_G^{-1}$, we have that $\lambda_i(\mathcal{L}_H) = \Theta(\lambda_i(\mathcal{L}_G))$, and the assumption of Υ in H is preserved from G up to a constant factor. By Lemma B.2 in Appendix B.1, the output of a spectral clustering algorithm on H satisfies the claimed properties. The total communication cost of $O(ns)$ words follows from the fact that every H_i has $O(n)$ edges. \square

Our proposed algorithm is very easy to implement, and the next theorem shows that the communication cost of our algorithm is optimal up to a logarithmic factor. The proof is given in Appendix B.2.

Theorem 3.2. *Let G be an undirected graph with n vertices, and the edges of G are distributed among s sites. Then, any algorithm that correctly outputs a constant fraction of a cluster in G requires $\Omega(ns)$ bits of communication cost. This holds even if each cluster has constant expansion.*

As a remark, it is easy to see that this lower bound also holds for constructing spectral sparsifiers: for any $n \times n$ PSD matrix A whose entries are arbitrarily distributed among s sites, any distributed algorithm that constructs a $(1 + \Theta(1))$ -spectral sparsifier of A requires $\Omega(ns)$ bits of communication. This follows since such a spectral sparsifier can be used to solve the spectral clustering problem. Spectral sparsification has played an important role in designing fast algorithms from different areas, e.g. machine learning, and numerical linear algebra. Hence our lower bound result for constructing spectral sparsifiers may have applications in studying other distributed learning algorithms.

3.2 The blackboard model

Next we present a graph clustering algorithm with $\tilde{O}(n + s)$ bits of communication cost in the blackboard model. We already saw from the proof of Theorem 3.1 that a spectral sparsifier preserves the cluster structure of a graph, so it suffices to present a distributed algorithm for constructing a spectral sparsifier in the blackboard model.

Our distributed algorithm for constructing a spectral sparsifier is based on constructing a chain of coarse sparsifiers [17], which is described as follows: for any input PSD matrix K with $\lambda_{\max}(K) \leq \lambda_u$ and all the non-zero eigenvalues of K at least λ_ℓ , we construct a chain of matrices

$$[K(0), K(1), \dots, K(d), K] \quad (3)$$

of length $d = \lceil \log_2(\lambda_u/\lambda_\ell) \rceil$, where $K(i) = K + \gamma(i)I$, and $\gamma(i) = \lambda_u/2^i$. Notice that in the chain (3) every $K(i-1)$ is obtained by adding weights to the diagonal entries of $K(i)$, and $K(i-1)$ approximates $K(i)$ as long as the weights added to the diagonal entries are small. Such a chain can be constructed recursively so that $K(0)$ has heavy diagonal entries, and can be approximated by a diagonal matrix. Moreover, it is easy to prove that $d = O(\log n)$ if K is the Laplacian matrix of a graph G , provided G 's edge weights are polynomially bounded.

Lemma 3.3 ([17]). *The chain (3) satisfies the following relations: (1) $K \preceq_r K(d) \preceq_r 2K$; (2) $K(\ell) \preceq K(\ell-1) \preceq 2K(\ell)$ for all $\ell \in \{1, \dots, d\}$; (3) $K(0) \preceq 2\gamma(0)I \preceq 2K(0)$.*

Based on Lemma 3.3, to construct a spectral sparsifier of K we build a chain of matrices

$$[\tilde{K}(0), \tilde{K}(1), \dots, \tilde{K}(d)], \quad (4)$$

such that every $\tilde{K}(\ell+1)$ can be constructed from $\tilde{K}(\ell)$, and every $\tilde{K}(\ell)$ is a spectral sparsifier of $K(\ell)$. It is well known that, for $K(\ell) \triangleq B^\top B$, sampling rows b_i of B according to their leverage scores, denoted by $\tau_i \triangleq b_i^\top K b_i$, will give a matrix approximating K . Formally, we assume that $\tilde{\tau}$ is the vector of leverage score overestimate for B 's rows such that $\tilde{\tau}_i \geq \tau_i$ for all $i \in [m]$, $0 < \varepsilon < 1$, and c is a fixed constant. We sample every row b_i with probability $p_i = \min\{1, c\varepsilon^{-2}\tilde{\tau}_i \log n\}$, and define a diagonal matrix W with $W_{i,i} = \frac{1}{p_i}$ with probability p_i , and $W(i,i) = 0$ otherwise. Then, it holds with high probability that $(1 - \varepsilon)K(\ell) \preceq \tilde{K}(\ell) = B^\top W B \preceq (1 + \varepsilon)K(\ell)$. Moreover, W has $O(\|\tilde{\tau}\|_1 \varepsilon^{-2} \log n)$ non-zeros with high probability. We prove in Appendix B.3 that the above sampling procedure can be implemented in the blackboard model, and this gives the following result:

Theorem 3.4. *Let G be an undirected graph of n vertices, where edges of G are allocated in s sites. Then, a spectral sparsifier of G can be constructed with total communication cost $\tilde{O}(n + s)$ in the blackboard model. That is, the chain (4) can be constructed with communication cost $\tilde{O}(n + s)$ in the blackboard model.*

Combining Theorem 3.4 and the proof of Theorem 3.1, we obtain a distributed algorithm in the blackboard model with total communication cost $\tilde{O}(n + s)$, and the performance of our algorithm is the same as the statement of Theorem 3.1. Notice that $\Omega(n + s)$ bits of communication are needed for graph clustering in the blackboard model, since the output of a clustering algorithm contains $\Omega(n)$ bits of information and each server needs to communicate at least one bit. Hence the communication cost of our proposed algorithm is optimal up to a poly-logarithmic factor.

4 Distributed geometric clustering

We now consider geometric clustering, including k -median, k -means and k -center. Let P be a set of points of size n in a metric space with distance function $d(\cdot, \cdot)$, and let $k \leq n$ be an integer. In k -center we want to find a set C ($|C| = k$) such that $\max_{p \in P} d(p, C)$ is minimized, where $d(p, C) = \min_{c \in C} d(p, c)$. In k -median and k -means we replace the objective function $\max_{p \in P} d(p, C)$ with $\sum_{p \in P} d(p, C)$ and $\sum_{p \in P} (d(p, C))^2$ respectively.

4.1 The message passing model

As mentioned, for constant dimensional Euclidean space and a constant $c > 1$, there are algorithms that c -approximate k -median and k -means using $\tilde{O}(sk)$ bits of communication [3]. For k -center, the folklore parallel guessing algorithms (see, e.g., [9]) achieve a 2.01 approximation using $\tilde{O}(sk)$ bits of communication.

Our proof is based on the following Multiparty Set-Disjointness problem ($\text{DISJ}_{s,n}$): for any s sites $\mathcal{P}_1, \dots, \mathcal{P}_s$, where each \mathcal{P}_i has a set $S_i \subseteq [n]$, let $X_i = (X_i^1, \dots, X_i^n)$ be the characteristic vector of S_i , and let $X = (X_1, \dots, X_s)$ be the input matrix with X_i being the i -th row. Let $X^j = (X_1^j, \dots, X_s^j)$ be the j -th column of the input matrix X . We define a function ALLONE_s on an s -bit vector $Y = (Y_1, \dots, Y_s)$ as $\text{ALLONE}_s(Y) = \bigwedge_{i \in [s]} Y_i$, and $\text{DISJ}_{s,n}(X) = \bigvee_{j \in [n]} \text{ALLONE}_s(X^j)$. Then the $\text{DISJ}_{s,n}$ problem asks for the value of $\text{DISJ}_{s,n}(X)$.

We now show that these bounds are tight up to logarithmic factors.

Theorem 4.1. *For any $c > 1$, computing c -approximation for k -median, k -means or k -center correctly with probability 0.99 in the message passing model needs $\Omega(sk)$ bits of communication.*

Proof. We prove a more general results: the $\Omega(sk)$ lower bound holds for any *eligible* function which evaluates 0 if there are at most k points, and evaluates greater than 0 if there are at least $k + 1$ points. Note that k -median, k -means and k -center are all eligible functions. We prove this by a simple reduction from $\text{DISJ}_{s,\ell}$ where $\ell = (k + 1)/2$ (w.l.o.g., assuming k is odd). The reduction is as follows. Given an s -player set-disjointness instance of size ℓ (i.e., $\text{DISJ}_{s,\ell}$), let $X_i = (X_i^1, \dots, X_i^\ell)$ be the i -th row of the input matrix X . Let p^1, \dots, p^ℓ and q^1, \dots, q^ℓ be 2ℓ distinct point locations on a line. Each site i does the following: for each coordinate j , if $X_i^j = 0$ then it put a point u_i^j at location q^j ; otherwise if $X_i^j = 1$ it put a point at location p^j . It is easy to see that $\text{DISJ}_{s,\ell} = 1$ iff the number of distinct points in $\bigcup_{i \in [s], j \in [\ell]} u_i^j$ is $2(\ell - 1) + 1 = k$; and $\text{DISJ}_{s,\ell} = 0$ iff the number of distinct points in $\bigcup_{i \in [s], j \in [\ell]} u_i^j$ is $2(\ell - 1) + 2 = k + 1$. The lower bound follows from the definition of eligible function and Theorem B.3 in Appendix B.2. \square

A number of works on clustering consider *bicriteria* solutions (e.g., [12, 6]). An algorithm is an (c_1, c_2) -approximation ($c_1, c_2 > 1$) if the optimal solution costs W when using k centers, then the output of the algorithm costs at most $c_1 W$ when using at most $c_2 k$ centers. We can show that for k -median and k -means, the $\Omega(sk)$ lower bound holds even for algorithms with bicriteria approximations. The proof of the following theorem can be found in Appendix C.1. In this proof we use a finer notion of communication complexity called *information complexity*; see Appendix A for details.

Theorem 4.2. *For any $c \in [1, 1.01]$, computing $(7.1 - 6c, c)$ -bicriteria-approximation for k -median or k -means correctly with probability 0.99 in the message passing model needs $\Omega(sk)$ bits of communication.*

4.2 The blackboard model

In Appendix C.2 we show that there is an algorithm that achieves an $O(1)$ -approximation using $\tilde{O}(s + k)$ bits of communication for k -median and k -means. For k -center, it is straightforward to implement the parallel guessing algorithm in the blackboard model using $\tilde{O}(s + k)$ bits of communication.

Theorem 4.3. *There are algorithms that compute $O(1)$ -approximations for k -median, k -means and k -center correctly with probability 0.9 in the blackboard model using $\tilde{O}(sk)$ bits of communication.*

5 Experiments

In this section we present experimental results for spectral graph clustering in the message passing and blackboard models. We will compare the following three algorithms. (1) **Baseline**: each site sends all the data to the coordinator directly; (2) **MsgPassing**: our algorithm in the message passing model (Section 3.1); (3) **Blackboard**: our algorithm in the the blackboard model (Section 3.2).

Besides giving the visualized results of these algorithms on various datasets, we also measure the qualities of the results via the *normalized cut*, defined as $\text{ncut}(A_1, \dots, A_s) = \frac{1}{2} \sum_{i \in [s]} \frac{W(A_i, V \setminus A_i)}{\text{vol}(A_i)}$, which is the objective function that the original spectral clustering algorithm wants to minimize.

We implemented the algorithms using multiple languages, including Matlab, Python and C++. Our experiments were conducted on an IBM NeXtScale nx360 M4 server, which is equipped with 2 Intel Xeon E5-2652 v2 8-core processors, 32GB RAM and 250GB local storage.

Datasets. We test the algorithms in the following real and synthetic datasets, whose visualizations are presented in Figure 4 in Appendix D.1.

- **Twomoons**: this dataset contains $n = 14,000$ coordinates in \mathbb{R}^2 . We consider each point as a vertex. For any two vertices u, v , we add an edge with weight $w_{uv} = \exp\{-\|u - v\|_2^2 / \sigma^2\}$ with $\sigma = 0.1$ when one vertex is among the 7000-nearest points of the other. This construction results in a graph with about 110,000,000 edges.
- **Gauss**: this dataset contains $n = 10,000$ points in \mathbb{R}^2 . There are 4 clusters in this dataset, each generated using a Gaussian distribution. We construct a complete graph as the similarity graph. For any two vertices u, v , we define the weight $w_{uv} = \exp\{-\|u - v\|_2^2 / \sigma^2\}$ with $\sigma = 1$. The resulting graph has about 100,000,000 edges.
- **Sculpture**: a photo of *The Greek Slave*¹. We use an 80×150 version of this photo where each pixel is viewed as a vertex. To construct a similarity graph, we map each pixel to a point in \mathbb{R}^5 , i.e., (x, y, r, g, b) , where the latter three coordinates are the RGB values. For any two vertices u, v , we then put an edge between u, v with weight $w_{uv} = \exp\{-\|u - v\|_2^2 / \sigma^2\}$ with $\sigma = 0.5$ if one of u, v is among the 5000-nearest points of the other. This results in a graph with about 70,000,000 edges.

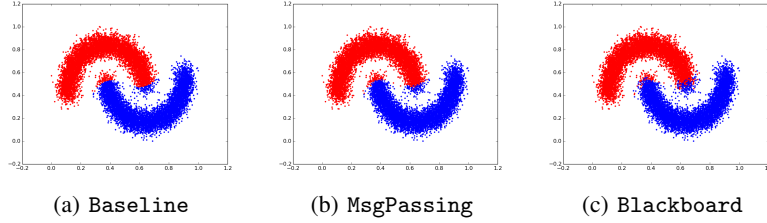
In the distributed model edges are randomly partitioned across s sites.

Results on Clustering Quality. We visualize the clustered results for different datasets in Figure 1 and Figure 5 (in Appendix D.2). It can be seen that **Baseline**, **MsgPassing** and **Blackboard** give results of very similar qualities. We only present the visualization for $s = 15$. Similar results were observed when we varied the values of s .

We also compare the normalized cut (ncut) values of the clustering results of different algorithms. The results are presented in Figure 2. In all datasets, the ncut values of different algorithms are very close. The ncut value of **MsgPassing** slightly decreases when we increase the value of s , while the ncut value of **Blackboard** is independent of s .

Results on Communication Costs. We compare the communication costs of different algorithms in Figure 3. We observe that while achieving similar clustering qualities as **Baseline**, both **MsgPassing** and **Blackboard** are significantly more communication-efficient (by one or two orders of magnitudes in our experiments). We also notice that the value of s does not affect the communication cost of **Blackboard**, while the communication cost of **MsgPassing** grows almost linearly with

¹Available in e.g. <http://artgallery.yale.edu/collections/objects/14794>



Twomoons, $k = 2$;

Figure 1: visualization of results on Twomoons; in the message passing model each site samples $5n$ edges; in the blackboard model all sites jointly sample $10n$ edges and the chain has length 18.

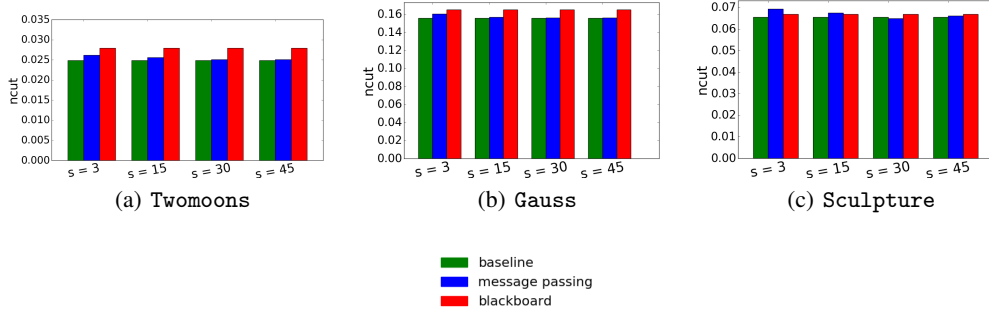


Figure 2: Comparisons on normalized cuts. In the message passing model, each site samples $5n$ edges; in each round of the algorithm in the blackboard model, all sites jointly sample $10n$ edges (in Twomoons and Gauss) or $20n$ edges (in Sculpture) edges and the chain has length 18.

s ; when s is large, MsgPassing uses significantly more communication than Blackboard. These confirm our theory. In Figure 6 and Figure 7 in Appendix D.3 we present how the performance of MsgPassing and Blackboard are affected by their parameters.

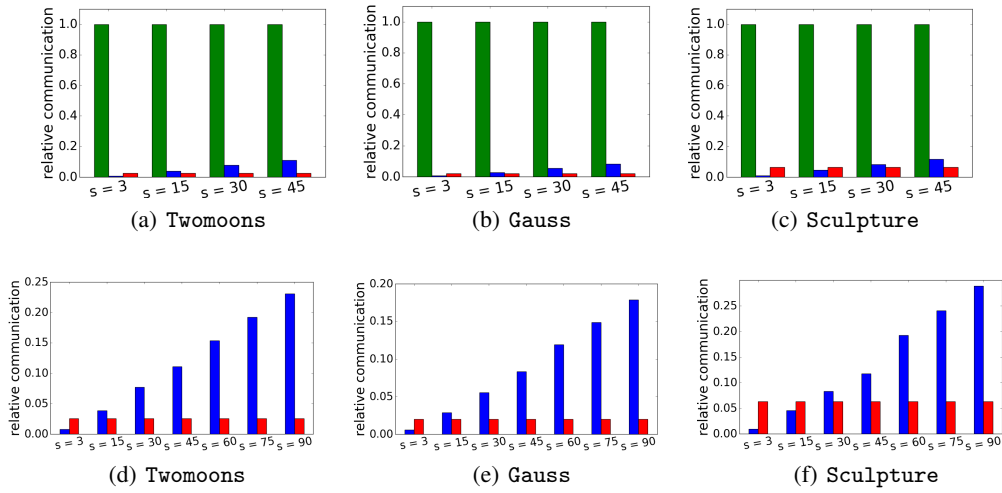


Figure 3: Comparisons on communication costs. In the message passing model, each site samples $5n$ edges; in each round of the algorithm in the blackboard model, all sites jointly sample $10n$ (in Twomoons and Gauss) or $20n$ (in Sculpture) edges and the chain has length 18.

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A Omitted details from Section 2

In this section we list some omitted definitions that will be used in our analysis.

Communication complexity. For any problem \mathcal{A} and any protocol Π solving \mathcal{A} , the *communication complexity* of a protocol Π is the maximum communication cost of Π over all possible inputs X . When the protocol is randomised, we define the *error* of Π by

$$\max_X \mathbb{P}(\text{the coordinator outputs an incorrect answer on } X),$$

where the max is over all inputs X and the probability is over all random strings of the coordinator and s sites. The δ -error randomised communication complexity $R_\delta(\mathcal{A})$ of a problem \mathcal{A} in the message passing model is the minimum communication complexity of any randomised protocol Π that solves \mathcal{A} with error at most δ .

Let μ be an input distribution on X . We call a deterministic protocol (δ, μ) -error if it gives the correct answer for \mathcal{A} on at least a $1 - \delta$ fraction of all input pairs, weighted by the distribution μ . We denote $D_{\delta, \mu}(\mathcal{A})$ as the cost of the minimum-communication (δ, μ) -error protocol. A standard lemma in communication complexity called Yao's minimax lemma shows that $R_\delta(\mathcal{A}) \geq \max_\mu D_{\delta, \mu}(\mathcal{A})$.

Information complexity. We abuse notation by using Π for both the protocol and its transcript (its concatenation of messages). In the message passing model, let Π_i ($i \in [s]$) be the transcript (set of messages exchanged) between the i -th site and the coordinator. Then Π can be seen as a concatenation $\Pi_1 \circ \Pi_2 \circ \dots \circ \Pi_s$ ordered by the timestamps of the messages. We define the information complexity of a problem \mathcal{A} in the message passing model by

$$\text{IC}_{\mu, \delta}(\mathcal{A}) = \min_{(\delta, \mu)\text{-error } \Pi} \sum_{i \in [s]} I(X_1, \dots, X_s; \Pi_i).$$

It has been shown in [11] that $R_\delta(\mathcal{A}) \geq \text{IC}_{\delta, \mu}(\mathcal{A})$ for any input distribution μ .

B Omitted details from from Section 3

This section lists all omitted details from Section 3.

B.1 Omitted details used in proving Theorem 3.1

Now we list all omitted details used in proving Theorem 3.1. We first show that a spectral sparsifier of G preserves the cluster structure.

Lemma B.1. *Let H be a $(1 + \varepsilon)$ -spectral sparsifier of G for some $\varepsilon \leq 1/3$. Then, it holds for any set $S \subseteq V$ that $\phi_H(S) \in (\frac{1}{2}, 2) \phi_G(S)$.*

Proof. Let $x_u \in \mathbb{R}^n$ be the indicator vector of vertex u , i.e., $x_u(v) = 1$ if $u = v$, and $x_u(v) = 0$ otherwise. We have that

$$(1 - \varepsilon) \cdot x_u^\top L_G x_u \leq x_u^\top L_H x_u \leq (1 + \varepsilon) \cdot x_u^\top L_G x_u,$$

which implies that $(1 - \varepsilon) \cdot \text{vol}_G(S) \leq \text{vol}_H(S) \leq (1 + \varepsilon) \cdot \text{vol}_G(S)$ for any subset S .

Similarly, for any set $S \subseteq V$ we define the indicator vector of S by $x_S \in \mathbb{R}^n$, where $x_S(u) = 1$ if $u \in S$, and $x_S(u) = 0$ otherwise. Hence, $x_S^\top L_G x_S = w_G(S, V \setminus S)$, and $x_S^\top L_H x_S = w_H(S, V \setminus S)$. Combining these with (1), we have that

$$(1 - \varepsilon) \cdot w_G(S, V \setminus S) \leq w_H(S, V \setminus S) \leq (1 + \varepsilon) \cdot w_G(S, V \setminus S).$$

Hence, for any subset S we have that

$$\phi_H(S) = \frac{w_H(S, V \setminus S)}{\text{vol}_H(S)} \leq \frac{(1 + \varepsilon)w_G(S, V \setminus S)}{(1 - \varepsilon) \text{vol}_G(S)} \leq 2 \cdot \phi_G(S),$$

where the last inequality holds by assuming $\varepsilon \leq 1/3$. Similarly, we have that

$$\phi_H(S) = \frac{w_H(S, V \setminus S)}{\text{vol}_H(S)} \geq \frac{(1 - \varepsilon)w_G(S, V \setminus S)}{(1 + \varepsilon) \text{vol}_G(S)} \geq \frac{1}{2} \cdot \phi_G(S).$$

Hence, $\phi_H(S_i)$ and $\phi_G(S_i)$ differ by at most a factor of 2. \square

Our analysis on the performance of spectral clustering relies on the following theorem.

Lemma B.2 ([19]). *Let G be a graph satisfying the condition $\Upsilon = \Omega(k^3)$, and $k \in \mathbb{N}$. Then, a spectral clustering algorithm outputs sets A_1, \dots, A_k such that $\text{vol}(A_i \triangle S_i) = O(k^3 \cdot \Upsilon^{-1} \cdot \text{vol}(S_i))$ holds for any $1 \leq i \leq k$, where S_i is the optimal cluster corresponding to A_i .*

B.2 Omitted details used in proving Theorem 3.2

Now we show a lower bound on the communication complexity of graph clustering in the message passing model. Our proof is based on a reduction from graph clustering to the Multiparty Set-Disjointness problem ($\text{DISJ}_{s,n}$): for any s sites $\mathcal{P}_1, \dots, \mathcal{P}_s$, where each \mathcal{P}_i has a set $S_i \subseteq [n]$, let $X_i = (X_i^1, \dots, X_i^n)$ be the characteristic vector of S_i , and let $X = (X_1, \dots, X_s)$ be the input matrix with X_i being the i -th row. Let $X^j = (X_1^j, \dots, X_s^j)$ be the j -th column of the input matrix X . We define a function ALLONE_s on an s -bit vector $Y = (Y_1, \dots, Y_s)$ as $\text{ALLONE}_s(Y) = \bigwedge_{i \in [s]} Y_i$, and $\text{DISJ}_{s,n}(X) = \bigvee_{j \in [n]} \text{ALLONE}_s(X^j)$. Then the $\text{DISJ}_{s,n}$ problem asks the value of $\text{DISJ}_{s,n}(X)$. We introduce two hard input distributions for ALLONE_s and $\text{DISJ}_{s,n}$ respectively.

- *Hard input distribution ν on $Y \in \{0, 1\}^s$ for ALLONE_s :* with probability $1/2$, we choose each Y_i ($i \in [s]$) to be 0 or 1 with equal probability; with probability $1/4$ we choose Y to be an all-1 vector; and with the rest probability $1/4$ we choose Y to be a random vector with $n-1$ coordinates being 1's and a random coordinate being 0.
- *Hard input distribution μ_n on $X \in \{0, 1\}^{s \times n}$ for $\text{DISJ}_{s,n}$:* For each $j \in [n]$, we choose $X^j \sim \nu$.

Theorem B.3 ([5]). *It holds that $\text{IC}_{0.49, \nu}(\text{ALLONE}_s) = \Omega(s)$, and $\text{IC}_{0.49, \nu}(\text{DISJ}_{s,n}) = \Omega(sn)$.*

Lemma B.4. *In the message passing model, any randomized algorithm that computes $\text{DISJ}_{s,n}$ correctly with probability 0.9 needs $\Omega(sn)$ bits of communication.*

Proof. The lemma follows from Theorem B.3 and Yao's minimax lemma. \square

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Our proof is based on the reduction from graph clustering to the Multiparty Set-Disjointness problem ($\text{DISJ}_{s,n}$). For any item j and site \mathcal{P}_i , we set $X_i^j = 0$ if item j appears in site \mathcal{P}_i , and $X_i^j = 1$ otherwise. Then $\text{DISJ}_{s,n}(X) = 1$ if there is some item not appearing in any site. Now we construct a graph G based on the hard instance X of $\text{DISJ}_{s,n}$ as follows: initially, graph G consists of n isolated vertices ℓ_1, \dots, ℓ_n , and s isolated vertices r_1, \dots, r_s . Then, we add an edge between ℓ_j and r_i if item j appears in site \mathcal{P}_i . With this construction, it is easy to see that $\text{DISJ}_{s,n}(X) = 0$ if every vertex ℓ_j is connected to some r_i , and $\text{DISJ}_{s,n}(X) = 1$ if there are some isolated vertices ℓ_j .

We will show that, when $\text{DISJ}_{s,n}(X) = 0$, our constructed graph G is a bipartite expander, i.e., G has only 1 cluster. To prove this, notice that, from the hard input distribution μ on $Y \in \{0, 1\}^s$ described above, with probability $1/2$ we choose each Y_i ($i \in [s]$) to be 0 or 1 with equal probability. This implies that, for any ℓ_i and r_j , there is an edge between ℓ_i and r_j independently with probability at least $1/4$. By standard results on constructing expanders, this implies G is a bipartite expander with constant expansion, and in particular is connected.

On the other side, when $\text{DISJ}_{s,n}(X) = 1$, every isolated vertex ℓ_j itself forms a cluster with conductance 0 and constant expansion, and the giant component of G forms a cluster with conductance 0 and constant expansion (since, as argued in the previous paragraph, it is a bipartite expander). Let k be the number of connected components in graph G . Then, $\rho(k) = 0$, and our assumption on $\Upsilon = \lambda_{k+1}(\mathcal{L}_G)/\rho(k) = \Omega(k^3)$ holds trivially. Hence any clustering algorithm that is able to find a constant fraction of each cluster in graph G satisfying $\Upsilon = \Omega(k^3)$ can be used to solve $\text{DISJ}_{s,n}$, the lower bound of communication complexity of graph clustering follows from the lower bound for $\text{DISJ}_{s,n}$. \square

B.3 Proof of Theorem 3.4

Proof of Theorem 3.4. Let $K = B^\top B$ be the Laplacian matrix of the underlying graph G , where $B \in \mathbb{R}^{m \times n}$ is the edge vertex incident matrix of G . We will prove that every $\tilde{K}(i+1)$ can be constructed based on $\tilde{K}(i)$ with communication cost $\tilde{O}(n+s)$. This implies that $\tilde{K}(d)$, a $(1+\varepsilon)$ -spectral sparsifier of K , can be constructed with communication cost $\tilde{O}(n+s)$, as the length of the chain $d = O(\log n)$.

First of all, notice that $\gamma_u \leq 2n$, and the value of n can be obtained with communication cost $\tilde{O}(n+s)$ (different sites sequentially write the new IDs of the vertices on the blackboard). For simplicity, in the following we assume that γ_u is the upper bound of λ_{\max} that we actually obtained in the blackboard model.

Base case of $\ell = 0$: By definition, $K(0) = K + \lambda_u \cdot I$, and

$$\frac{1}{2} \cdot K(0) \preceq \gamma(0) \cdot I \preceq K(0),$$

due to Statement 3 of Lemma 3.3. Letting \oplus denote appending the rows of one matrix to another, we define $B_{\gamma(0)} = B \oplus \sqrt{\gamma(0)} \cdot I$, and write $K(0) = K + \gamma(0) \cdot I = B_{\gamma(0)}^\top B_{\gamma(0)}$. Since $\tau_i = b_i^\top (K(0))^\top b_i$ for each row of $B_{\gamma(0)}$, we have

$$\tau_i \leq b_i^\top (\gamma(0) \cdot I) b_i \leq 2 \cdot \tau_i.$$

Let $\tilde{\tau}_i = b_i^\top (\gamma(0) \cdot I)^+ b_i$ be the leverage score of b_i approximated using $\gamma(0) \cdot I$, and let $\tilde{\tau}$ be the vector of approximate leverage score, with the leverage scores of the n rows corresponding to $\sqrt{\gamma(0)} \cdot I$ rounded up to 1. Then, with high probability sampling $O(\varepsilon^{-2} n \log n)$ rows of B will give a matrix $\tilde{K}(0)$ such that

$$(1 - \varepsilon)K(0) \preceq \tilde{K}(0) \preceq (1 + \varepsilon)K(0).$$

Notice that, as every row of B corresponds to an edge of G , the approximate leverage scores $\tilde{\tau}_i$ for different edges can be computed locally by different sites maintaining the edges, and the sites only need to send the information of the sampled edges to the coordinator, hence the communication cost is $\tilde{O}(n+s)$.

Induction step: We assume that

$$(1 - \varepsilon)K(\ell) \preceq_r \tilde{K}(\ell) \preceq_r (1 + \varepsilon)K(\ell),$$

and the coordinator maintains the matrix $\tilde{K}(\ell)$. This implies that

$$\frac{1 - \varepsilon}{1 + \varepsilon} K(\ell) \preceq_r \frac{1}{1 + \varepsilon} \tilde{K}(\ell) \preceq_r K(\ell).$$

Combining this with Statement 2 of Lemma 3.3, we have that

$$\frac{1 - \varepsilon}{2(1 + \varepsilon)} K(\ell + 1) \preceq_r \frac{1}{2(1 + \varepsilon)} \tilde{K}(\ell) \preceq K(\ell + 1).$$

We apply the same sampling procedure as the base case, and obtain a matrix $\tilde{K}(\ell + 1)$ such that

$$(1 - \varepsilon)K(\ell + 1) \preceq_r \tilde{K}(\ell + 1) \preceq_r (1 + \varepsilon)K(\ell + 1).$$

Notice that, given $\tilde{K}(\ell)$ in the blackboard, the probabilities used for sampling individual edges can be computed locally by different sites, and in each round only the sampled edges will be sent to the coordinator in order for the coordinator to obtain $\tilde{K}(\ell + 1)$. Hence, the total communication cost in each iteration is $\tilde{O}(n+s)$. \square

C Omitted details from Section 4

C.1 Proof of Theorem 4.2

Proof. We can show the following technical lemma.

Lemma C.1. *In the message-passing model, $\Omega(s\ell)$ bits of communication is needed for computing at least a 0.8 fraction of $j \in [\ell]$ $\text{ALLONE}_s(X^j)$ correctly with probability 0.99 under the input distribution $X \sim \mu_\ell$.*

Proof. By a Markov inequality, there must exist $\Omega(s)$ coordinates j such that the algorithm computes $\text{ALLONE}_s(X^j)$ ($X^j \sim \nu$) with error probability at most 0.24. Call each of such coordinates j *good*. Let Π be the protocol transcript. We have

$$\begin{aligned} I(X; \Pi) &= \sum_{j \in [\ell]} I(X^j; \Pi \mid X^{-j}) \\ &\geq \sum_{j \in [\ell]} I(X^j; \Pi) \quad (X^j \text{ and } X^{-j} \text{ are independent}) \\ &\geq \sum_{\text{good } j} I(X^j; \Pi) \\ &\geq \Omega(s) \cdot \text{IC}_{0.24, \nu}(\text{ALLONE}_s) \\ &\geq \Omega(s\ell). \quad (\text{Theorem B.3}) \end{aligned}$$

□

The reduction. We consider 8ℓ point locations on a line with x -coordinates being $1, 2, \dots, 8\ell$. We put a point with *infinite* weight at every even point location. We name the 4ℓ odd point locations from left to right as

$$p^1, q^1, p^2, q^2, \dots, p^\ell, q^\ell, z^1, z^2, \dots, z^{2\ell}.$$

For each site $i \in [s]$ and each column $j \in [\ell]$, if $X_i^j = 0$ then we put a point with weight 1 at location q^j ; otherwise if $X_i^j = 1$ then we put a point with weight 1 at location p^j . We also put a point with weight $1/2$ at each of the “dummy” locations $z^1, \dots, z^{2\ell}$. Let the weight of a *location* be the sum of the weights of points falling into that location.

Given such an input X , for both k -median and k -means, the optimal solution (OPT) which is allowed to use $k = 6\ell$ centers will include all locations p^j and q^j whose weights are at least 1 (note that there are *at most* 2ℓ such locations), the 4ℓ even point locations, and as many as dummy locations that it can still include. The cost of the optimal solution will be precisely the cost of linking the points in the rest of the dummy locations to their nearest centers (at the even locations), which can be written as

$$\text{OPT} = 1/2 \cdot (k/3 - (k/3 - F_0)) = 1/2 \cdot F_0 \leq \ell,$$

where F_0 is the number of locations in $\{p^1, q^1, \dots, p^\ell, q^\ell\}$ that have weights at least 1.

Now suppose our solution (SOL) outputs ck centers for a constant $c \in [1, 1.01]$. Each time we include a location q^j as a center when there is no 0-coordinate in the input column X^j , we have a loss of $1/2$ since we miss out on including a dummy location (i.e., we can take one more dummy location instead of taking q^j as a center). Similarly, each time we do not include a location q^j as a center when there is a 0-coordinate in X^j , we have a loss of $1/2$ since a point at q^j has weight at least 1 but a point at a dummy location has weight at $1/2$. Therefore, even if we are allowed to output ck medians, we will still need to figure out whether there is any point at location q^j for at least an $\alpha = 0.9$ fraction of the coordinates $j \in [\ell]$. If not, then

$$\begin{aligned} \text{SOL} - \text{OPT} &\geq 1/2 \cdot (1 - \alpha)\ell - 1 \cdot (c - 1)k \\ &= \frac{(1 - \alpha) - 12(c - 1)}{2} \cdot \ell \\ &\geq (6.1 - 6c)\text{OPT}, \end{aligned} \tag{5}$$

where the first term in the RHS of (5) counts the loss of incorrectly computing the (at least) $(1 - \alpha)\ell$ coordinates $j \in [\ell]$, and the second term counts the maximum gain of the extra $(c - 1)k$ centers SOL can use (compared with OPT).

By Lemma C.1, we have that for any $c \in [1, 1.01]$, computing $(7.1 - 6c, c)$ -bicriteria-approximation for k -median or k -means in the message passing model correctly with probability 0.9 under distribution $X \sim \mu$ needs $\Omega(sk)$ bits of communication. Theorem 4.2 follows by Yao’s minimax principle. □

C.2 Algorithm for geometric clustering in the blackboard model

Our algorithm for k -median/means is an easy adaptation of the *successive sampling* algorithm proposed by Mettu and Plaxton [16] in the (centralized) RAM model. We first summarize their algorithm and then describe how to port it to the blackboard model.

Let X_1, \dots, X_s be the point sets at sites $\mathcal{P}_1, \dots, \mathcal{P}_k$ respectively. The successive sampling algorithm proceeds in rounds. At each round j it does the following:

1. s sites jointly sample $O(k)$ point centers, denoted by Y_j ;
2. s sites grow balls from each of the point centers in Y_j synchronously until a time step when a 0.9 fraction of points in $\bigcup_{i \in [s]} X_i$ are covered;
3. each site \mathcal{P}_i updates X_j by removing those points that are covered by any of the balls centered at points in Y_j ;
4. s sites remove all the points covered by balls centered at points in Y_j , and proceed to the next round $j + 1$.

It is easy to see that the computation will finish in $r = O(\log n)$ rounds since at each round we remove a constant fraction of points. At the end we compute an $O(1)$ -approximation of k -median or k -means on the $O(k \log n)$ points $\bigcup_{j \in [r]} Y_j$. In [16] it has been shown that this algorithm gives an $O(1)$ -approximation to k -median or k -means with high probability.

We now describe how to implement this centralized algorithm in the blackboard model. We first consider each round. Step 1 can be done by the distributed sampling algorithm in [8] using $\tilde{O}(k + s)$ bits of communication; note that at the end of this step the sampled points in Y_j are written on the blackboard. Step 2 can be done by a binary search for the minimum ball radius t_j such that $\bigcup_{p \in Y_j} \text{Ball}(p, t_j)$ covers at least a 0.9 fraction of points in $\bigcup_{i \in [s]} X_i$, where $\text{Ball}(p, t_j)$ denotes the ball centered at p with radius t_j ; this binary search can be done using $\tilde{O}(1)$ bits of communication. Step 3 and 4 can be done locally without any communication. After r rounds, the final clustering step can be done by any of the s sites since all points in $\bigcup_{j \in [r]} Y_j$ have already been written on the blackboard.

D More Experimental Results

D.1 Datasets

Figure 4 visualizes the datasets for our experiments.

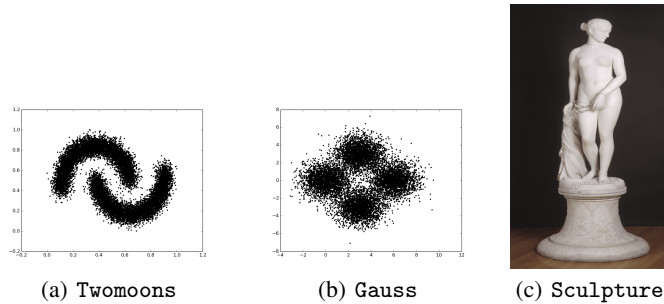


Figure 4: Visualization of the datasets for our experiments.

D.2 Quality of the Clustering

Figure 5 visualizes the qualities of the results returned by the three algorithms on datasets Gauss and Sculpture.

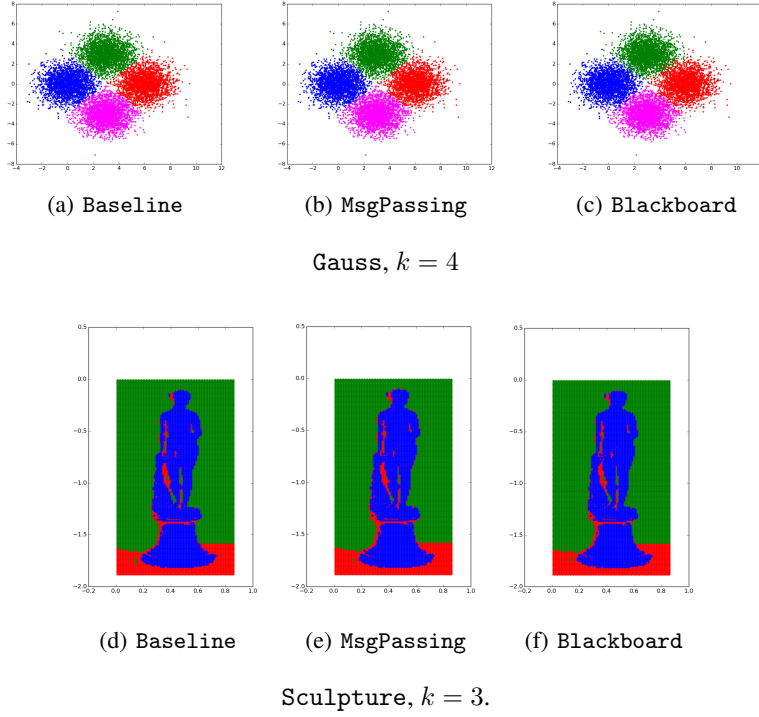
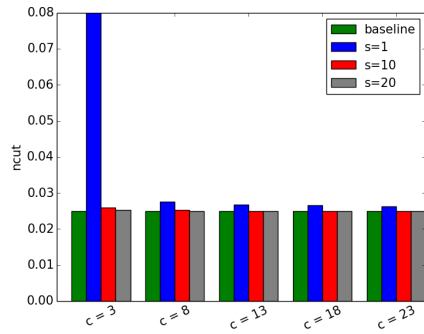


Figure 5: Visualization of results on Gauss and Sculpture; in the message passing model each site samples $5n$ edges; in the blackboard model all sites jointly sample $10n$ (in Gauss) or $20n$ (in Sculpture) edges and the chain has length 18.

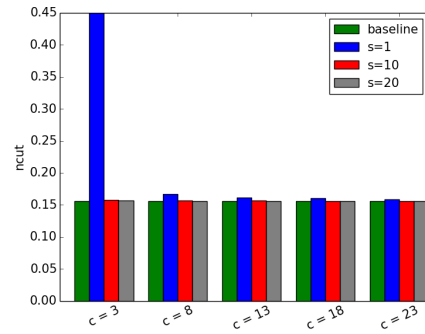
D.3 Parameters in MsgPassing and Blackboard

Figure 6 shows in MsgPassing, how the value of $ncut$ is affected by the number of sites and the number of edges sampled in each site. Here, each site samples cn edges. When $c = 3$ and $s = 1$, the $ncut$ value diverges in all datasets. This is because with such a small c , the algorithm does not generate a valid sparsifier. In general, increasing c or s will slightly decrease the $ncut$ value. But once they are above some thresholds, the $ncut$ values of MsgPassing and Baseline become very close.

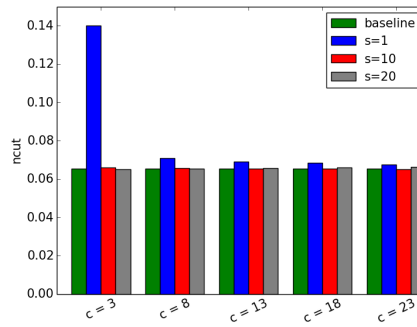
Figure 7 shows in Blackboard, how the $ncut$ value is affected by the number of iterations and the number of edges sampled. When the number of iterations is set to be 5, $ncut$ values diverge in all datasets. This is because we cannot expect to generate a valid sparsifier by using such few iterations. It can be seen from 7(b) that for a fixed c , performing more iterations will help to reduce $ncut$ values. From the same figure, one can also conclude that for fixed iterations, increasing c also help to reduce the $ncut$ values.



(a) Twomoons

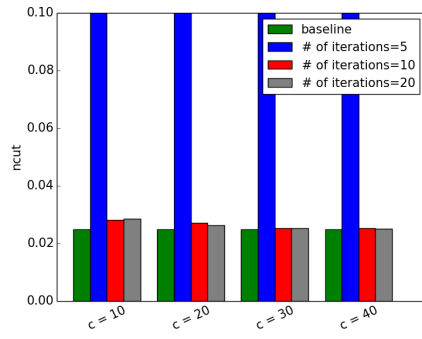


(b) Gauss dataset

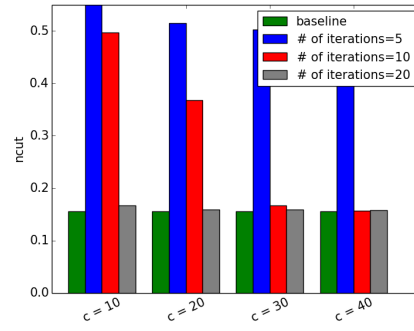


(c) Sculpture

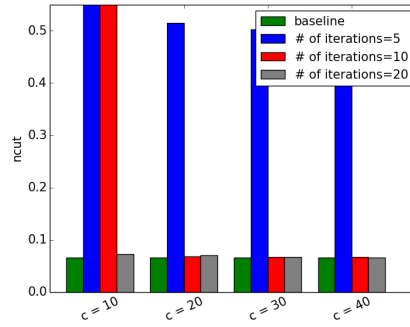
Figure 6: The pictures above show the $ncut$ values with respect to the values of c and s for the MsgPassing algorithm. Here each site samples cn edges.



(a) Twomoons



(b) Gauss



(c) Sculpture

Figure 7: The pictures above show how the ncut values are affected by the number of iterations and the value of c for the Blackboard algorithm. Here all sites jointly sample cn edges.